CITUSC/00-019 hep-th/0004090

2T Physics Formulation of Superconformal Dynamics Relating to Twistors and Supertwistors[1]

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Abstract

The conformal symmetry SO$(d,2)$ of the massless particle in d dimensions, or superconformal symmetry OSp$(N|4)$, SU$(2,2|N)$, OSp$(8|N)$ of the superparticle in $d=3,4,6$ dimensions respectively, had been previously understood as the global Lorentz symmetry and supersymmetries of 2T physics in $d+2$ dimensions. By utilizing the gauge symmetries of 2T physics, it is shown that the dynamics can be cast in terms of superspace coordinates, momenta and theta variables or in terms of supertwistor variables à la Penrose and Ferber. In 2T physics these can be gauge transformed to each other. In the supertwistor version the quantization of the model amounts to the well known oscillator formalism for non-compact supergroups.

1 2T Physics

Two-time (2T) physics has by now been shown to provide a reformulation of all possible one-time (1T) particle dynamics, including interactions with Yang-Mills, gravitational and other fields [1]-[8]. 2T physics has mainly been developed in the context of particles, but some advances have also been made with strings and p-branes [6], and some insights for M-theory have already emerged [5][9]. In the case of particles, there exists a general worldline formulation with background fields [7], as well as a field theory formulation [8], both described in terms of fields that depend on $d+2$ coordinates X^M. The 2T point of view has been useful in bringing new insights into 1T physics, first by revealing previously unnoticed hidden symmetries in 1T dynamical systems, and second by providing a unification of classes of 1T dynamics that are different in 1T physics, but

[1]This research was partially supported by the US. Department of Energy under grant number DE-FG03-84ER40168.

are regarded as being gauge equivalent to each other under the local gauge symmetries of a unique 2T system.

It is important to realize that the Standard Model of particle physics and gravity (i.e. all the physics we can verify) can be rewritten as a 2T field theory, modulo a reformulation of the mass term in the Higgs potential. The 2T approach is not a naive extension of the number of timelike dimensions, as this would be disastrous for unitarity and causality, but it does imply the existence of a more subtle higher structure of dimensional unification, including an extra timelike and an extra spacelike dimension, *plus gauge symmetries* that tend to reduce the number of effective dimensions (worldline $\mathrm{Sp}(2,R)$ or $\mathrm{OSp}(n|2)$, and local spacetime generalized bosonic/fermionic kappa symmetries). Thus, for a fixed set of 2T background fields (e.g. flat background) one can find many related interacting 1T systems by gauge fixing the underlying gauge symmetry. The non-trivial aspects of 2T physics is due to the fact that d dimensions (1T) can be embedded in many ways inside $d+2$ dimensions (2T) (also such concepts extended to superspace). The 1T observers see the same 2T system as different dynamics from the point of view of the chosen d dimensions, but all 1T observers are gauge equivalent from the point of view of the 2T observer. This gauge equivalence is translated to a kind of "duality" among the 1T dynamical systems that are derived from a given 2T system.

The origins of the field theory formalism goes back to Dirac's work in 1936 on conformal symmetry, but this approach historically was used exclusively as a reformulation of conformal symmetry in field theory [10]-[13]. An early approach on the worldline is also aimed to conformal symmetry [14]. The unification of different dynamics in the form of 2T physics was not understood until the work of [1], which reached the $\mathrm{Sp}(2,R)$ gauge theory formulation on the worldline by following a very different path and motivation (coming from 2T signals and dualities in M-theory, F-theory, S-theory, and influenced directly by the formalism in [15]). Also, in the field theory formalism, it was realized only recently [8] that the 2T field equations are a reflection of the $\mathrm{Sp}(2,R)$ gauge symmetries that underlie 2T physics on the worldline, and that the gauge fixing procedure which produce the various 1T dynamics in the worldline formulation can be carried out in the field theory formalism as well. Although the 2T unification of 1T systems can be examined in either the worldline or field theory forms, the worldline approach provides a better understanding of the underlying gauge symmetries at this stage, while the field theory formulation provides a familiar approach for interactions among fields.

In this paper we will further extend the gauge symmetries of 2T physics on the worldline by further developing some concepts already introduced in [4] that were necessary for the 2T formulation of supersymmetry. The gauge symmetry is associated with the spacetime $\mathrm{SO}(d,2)$, some internal symmetry and an extended concept of bosonic/fermionic kappa supersymmetry. We will use the $\mathrm{Sp}(2,R)$ gauge symmetry together with the extra gauge symmetry to show that particle dynamics in one of the $\mathrm{Sp}(2,R)$ 1T gauges (the $\mathrm{SO}(d-1,1)$ relativistic particle gauge[2]) can be cast either

[2]The same idea could be applied in other $\mathrm{Sp}(2,R)$ 1T gauges, but we will refrain from discussing other $\mathrm{Sp}(2,R)$ gauges in this paper.

in terms of particle coordinates and momenta (x^μ, p^μ) or in terms of twistor variables à la Penrose [16] or supertwistors à la Ferber [17]. This result will be extended to the conformal superparticle cases for $d = 3, 4, 6$ with N supersymmetries using the supergroups $\mathrm{OSp}(N|4)$, $\mathrm{SU}(2,2|N)$, $\mathrm{OSp}(8^*|2N)$ respectively. It will be shown that the (super)particle description (x, p, θ) is *gauge equivalent* to the (super)twistor description when the $\mathrm{SO}(d, 2)$ and $\mathrm{Sp}(2, R)$ gauge symmetries are utilized. Furthermore, it will be shown that the quantization of the model can be carried out in terms of oscillators associated with the (super)twistors, and the unitarity established. The main motivation for developing this formalism came from trying to understand the quantum theory of a toy M-model introduced in [5]. The methods presented here will be applied in [9] to the case of the toy M-model based on the supergroup $\mathrm{OSp}(1|64)$.

2 1T superparticle

Since our ultimate goal includes spacetime supersymmetry we will directly discuss the massless superparticle; the reader can delete the fermions to specialize to the purely bosonic case. The action for the superparticle in d dimensions, with N supersymmetries, is traditionally described by the following well known Lagrangian written in terms of position $x^\mu(\tau)$, momentum $p^\mu(\tau)$ and N spacetime spinors $\theta^a_\alpha(\tau)$ (α denotes the spinor and a=1,2,$\cdots$, N)

$$\pounds = \frac{1}{2A^{22}}\left(\dot{x}^\mu + \tilde{\theta}_a\gamma^\mu\partial_\tau\theta^a\right)^2 = \dot{x}\cdot p - \frac{1}{2}A^{22}p^2 + \tilde{\theta}_a\gamma\cdot p\partial_\tau\theta^a. \tag{1}$$

We used the symbol $A^{22}(\tau)$ for the einbein because of its relation to the $\mathrm{Sp}(2, R)$ gauge fields $A^{ij}(\tau)$ that we will see below. The general variation of the fields has the form (up to total derivatives)

$$\begin{aligned}\delta L &= \partial_\tau\left(\delta x - \delta\theta_a\gamma\theta^a\right)\cdot p + 2\delta\theta_a\left(p\cdot\gamma\right)\dot{\theta}^a &(2)\\ &\quad +\delta p\cdot\left(\dot{x}^\mu + \tilde{\theta}_a\gamma^\mu\partial_\tau\theta^a - A^{22}p\right) - \delta A^{22}\frac{p^2}{2} &(3)\end{aligned}$$

For general d and N, using the appropriate spinor, this Lagrangian is symmetric ($\delta L = 0$ up to total derivatives) under local τ-reparametrization, local kappa supersymmetry and global super Poincaré symmetry. In particular the global supersymmetry with parameters ε^a_α is given by

$$\delta_\varepsilon x^\mu = -\delta_\varepsilon\theta\gamma^\mu\theta,\quad \delta_\varepsilon\theta = \varepsilon,\quad \delta_\varepsilon p^\mu = 0,\quad \delta_\varepsilon A^{22} = 0, \tag{4}$$

and the local kappa transformation with parameters $\kappa^a_\alpha(\tau)$ is

$$\delta_\kappa x^\mu = \delta_\kappa\theta\gamma^\mu\theta,\quad \delta_\kappa\tilde{\theta}_a = \tilde{\kappa}_a\gamma\cdot p,\quad \delta_\kappa p^\mu = 0,\quad \delta_\kappa A^{22} = 4\tilde{\kappa}\dot{\theta}. \tag{5}$$

For the special dimensions $d = 3, 4, 6$ this Lagrangian is also invariant under dilations, conformal transformations and special superconformal transformations, such that the full global symmetry is given by the supergroups

$$G = OSp\left(N|4\right),\ SU\left(2,2|N\right),\ OSp\left(8^*|N\right) \tag{6}$$

respectively, as shown in [19] for $d=3,4$ and in [4] for $d=6$. The conformal subgroups in these dimensions are $\mathrm{SO}(3,2)=Sp(4)$, $\mathrm{SO}(4,2)=SU(2,2)$, and $\mathrm{SO}(6,2)=Spin(8^{*})$ respectively. For the purely bosonic case this Lagrangian has global conformal symmetry $\mathrm{SO}(d,2)$ for any d.

3 2T formulation

The conformal symmetry $\mathrm{SO}(d,2)$ is a giveaway for 2T physics. Indeed all of the above cases correspond to a special $\mathrm{Sp}(2,R)$ gauge choice of a 2T formulation (the $\mathrm{SO}(d-1,1)$ relativistic particle gauge) in which the $\mathrm{SO}(d,2)$ Lorentz symmetry in flat backgrounds in $d+2$ dimensions gets interpreted as the conformal symmetry in 1T (it acquires other interpretations in other $\mathrm{Sp}(2,R)$ gauges). The 2T reformulation of the superparticle in $d=3,4,6$ dimensions requires $d+2$ coordinates $X^{M}(\tau)$ and momenta $P^{M}(\tau)$, and a supergroup element $g(\tau)\in G$ that contains fermions $\Theta^{a}_{\tilde{\alpha}}(\tau)$ where $\tilde{\alpha}$ denotes the spinor in $d+2$ dimensions. This spinor has double the size of the spinor $\theta^{a}_{\alpha}(\tau)$ in d dimensions, which is of course necessary if the $\mathrm{SO}(d,2)$ is to be realized linearly in the 2T formulation. Thus, compared to the 1T formulation there are extra degrees of freedom in X,P,Θ and in the bosonic sectors in $g(\tau)$. If the covariant 2T formulation is to be equivalent to the 1T formulation there has to be various gauge symmetries and extended kappa supersymmetries to cut down the degrees of freedom to the correct set. Following [4] this is beautifully achieved as follows.

The 2T Lagrangian is

$$L=\dot{X}_1\cdot X_2-\frac{1}{2}A^{ij}X_i\cdot X_j-\frac{1}{s}Str\left(\Gamma_{MN}\dot{g}g^{-1}\right)L^{MN},\tag{7}$$

where $X_i^M=\left(X^M,P^M\right)$ is the $\mathrm{Sp}(2,R)$ doublet, A^{ij} is the $\mathrm{Sp}(2,R)$ gauge potential, Γ_M are gamma matrices and $\Gamma_{MN}=\frac{1}{2}\left[\Gamma_M,\Gamma_N\right]$ are the $\mathrm{SO}(d,2)$ generators in the spinor representation of dimension s, the Cartan connection $\dot{g}g^{-1}$ projected in the direction of the $\mathrm{SO}(d,2)\in G$ is coupled to the $\mathrm{SO}(d,2)$ orbital angular momentum $L^{MN}=\varepsilon^{ij}X_i^MX_j^N=X^MP^N-X^NP^M$ which is $\mathrm{Sp}(2,R)$ gauge invariant. Although our interest in this paper is on the supergroups G listed in (6) the discussion of local symmetries below applies also to any group or supergroup that contains $\mathrm{SO}(d,2)$ as a subgroup. In fact, for the toy M-model in [5][9] the case of interest is $G=OSp(1|64)$. In particular, for the purely bosonic particle one can simply take $G=SO(d,2)$.

The following is an improvement of the symmetry discussion given in [4]. From the extensive discussions in [1]-[9] we already know that the Lagrangian above has local symmetry under $\mathrm{Sp}(2,R)$. Beyond this, it obviously has global symmetry under G for the transformation of $g(\tau)$ by *right multiplication* by a global group element $g_R\in G$

$$X_i^M\rightarrow X_i^M,\quad A^{ij}\rightarrow A^{ij}\quad ,g\rightarrow gg_R.\tag{8}$$

Furthermore, it has local $\mathrm{SO}(d,2)$ Lorentz symmetry with parameters $\varepsilon^{MN}(\tau)$ under *left multi-*

plication of g in the spinor representation and transformation of X_i^M in the vector representation

$$\delta X_i^M = \varepsilon^{MN} X_{iN}, \quad \delta g = \frac{1}{4}\varepsilon^{MN}\left(\Gamma_{MN}\, g\right), \quad \delta A^{ij} = 0. \tag{9}$$

The time derivatives of $\dot{\varepsilon}^{MN}$ produced by the two kinetic terms in (7) cancel each other. Moreover, for the cases in which there is another subgroup in G (such as the $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, $\mathrm{Sp}(N)$ in (6)) with generators T_A that satisfy $\mathrm{Str}(T_A\Gamma_{MN}) = 0$, there is a local symmetry with parameters $\varepsilon^A(\tau)$ under *left multiplication* of g

$$\delta g = \varepsilon^A \left(T_A\, g\right), \quad \delta X_i^M = 0, \quad \delta A^{ij} = 0. \tag{10}$$

The time derivative $\dot{\varepsilon}^A$ as well other dependence on ε^A drops because $\mathrm{Str}(T_A\Gamma_{MN}) = 0$ and $[T_A, \Gamma_{MN}] = 0$. Finally there is a local bosonic/fermionic extended kappa (super)symmetry under *left multiplication* of g with infinitesimal coset elements K of the form (take $\mathrm{OSp}(N|2M)$ as an example for the matrix notation)

$$\delta g = Kg, \quad K = \begin{pmatrix} 0 & \xi(\tau) \\ \tilde{\xi}(\tau) & 0 \end{pmatrix}, \quad Str\left(\Gamma_{MN}K\right) = 0, \quad Str\left(T_A K\right) = 0, \tag{11}$$

provided δA^{ij} is non-zero as specified below, and $\xi_{\tilde{\alpha}}^{a}(\tau)$ has the form

$$\xi_{\tilde{\alpha}}^{a}(\tau) = X_i^M \left(\Gamma_M \kappa^{ia}\right)_{\tilde{\alpha}}\ , \tag{12}$$

where the local $\kappa_{\tilde{\alpha}}^{ia}(\tau)$ are unrestricted local parameters. Under such a transformation we have

$$\delta \pounds = 0 + \frac{2}{s} L^{MN} Str\left(\left[\Gamma_{MN}\,, K\right]\partial_\tau g g^{-1}\right) - \frac{1}{2}\delta A^{ij} X_i \cdot X_j,$$

One must have $L^{MN}(\Gamma_{MN}\xi)$ proportional to $X_i \cdot X_j$ so that δA^{ij} can be chosen to cancel the contribution from the first term. Indeed, the general form in (12) has this property

$$L^{MN}\left(\Gamma_{MN}\xi\right) = \varepsilon^{kj} X_k^M X_j^N X_i^R \left(\Gamma_{MN}\Gamma_R \kappa^i\right) = 2\varepsilon^{kj} X_k^M \left(\Gamma_M \kappa^i\right) X_j \cdot X_i \tag{13}$$

since the three gamma term in $\Gamma_{MN}\Gamma_R = \Gamma_{MNR} + \Gamma_M\eta_{NR} - \Gamma_N\eta_{MR}$ drops out due to the fact that the indices i, j, k take only two possible values.

Let us now specialize to a few cases of interest and use the gauge symmetries to cut down the degrees of freedom to those in d dimensions given in the beginning of the previous section.

- We start with the purely bosonic particle. Using the $\mathrm{SO}(d,2)$ local symmetry we can choose $g(\tau) = 1$ for all τ. The Lagrangian (7) is now expressed only in terms of X^M, P^M. We work in the basis $X^M = \left(X^{+'}, X^{-'}, x^\mu\right)$ where $X^{\pm'} = \left(X^{0'} \pm X^{1'}\right)/\sqrt{2}$ are lightcone type coordinates for the extra two dimensions (similarly for P^M). Using the $\mathrm{Sp}(2,R)$ local symmetry we can choose $X^{+'}(\tau) = 1$ and $P^{+'}(\tau) = 0$, and solve the two constraints $X^2 = 0$, $X \cdot P = 0$, to give $X^{-'} = x^2/2$ and $P^{-'} = x \cdot p$. The Lagrangian reduces to the bosonic particle Lagrangian

in (1) without the fermions. When $g=1$ the global SO$(d,2)$ that acts on the right of g must be compensated by a global transformation on the left of g which also acts on X_i^M. Thus the left/right transformations on g get coupled and become the 2T spacetime global SO$(d,2)$ Lorentz symmetry. When the Sp$(2,R)$ gauges are also fixed, this global symmetry acts non-linearly on the remaining variables (x^μ,p^μ) and is then interpreted as the conformal symmetry of the massless particle.

- Next we discuss the superparticle in $d=3,4,6$ as in [4]. Using the local SO$(d,2)$ and local internal symmetries the form of g, for G given in (6), can be gauge fixed to $g\rightarrow t=\exp\left(fermionic\ \ coset\right)$ parametrized by the spinor $\Theta_{\alpha}^{a}\left(\tau\right)$. This was the starting point in [4]. Using the kappa gauge $\Gamma^{+^{\prime}}\Theta=0$, and Sp$(2,R)$ gauges $X^{+^{\prime}}=1,\ \ P^{+^{\prime}}=0$ it was shown in [4] that the 2T Lagrangian reduces precisely to the superparticle Lagrangian in (1) for $d=3,4,6$. The global symmetry G becomes the non-linearly realized superconformal symmetry of the massless superparticle in these special dimensions.

- For other supergroups the use of the local symmetries can never reduce the degrees of freedom to only the superspace variables $\left(x,p,\theta\right)$. Generally there are more bosonic degrees of freedom. It was speculated in [4][5][9] that the extra degrees of freedom can be associated with collective coordinates that describe D-brane degrees of freedom in the particle limit. This is because the superalgebra has central extensions with relations among the charges such that BPS conditions are satisfied, as is the case for D-branes. This point will be further elaborated in [9] for the case of OSp$(1|64)$.

4 Twistors, supertwistors, oscillators

To make the presentation as explicit as possible we will concentrate on the $d=3$ superparticle, rewritten in the $d+2=5$ two-time formalism. Hence we will take $G=OSp\left(N|4\right)$ where Sp$(4)=SO\left(3,2\right)$. However, we will begin more generally with $G=$OSp$(M/2N)$ where Sp$(2N)$ has an SO$(d,2)$ subgroup whose spinor representation has dimension $s=2N$. For example for $d=11$ in the toy M-model, $SO\left(11,2\right)$ has a spinor with 64 components, and therefore we would consider OSp$(1|64)$.

In the basis $X^M=\left(X^{+^{\prime}},X^{-^{\prime}},X^{\mu}\right)$, we define the following SO$(d,2)$ gamma matrices

$$\Gamma^{\pm^{\prime}}=\pm\tau^{\pm}\times 1,\quad \Gamma^{\mu}=\tau_3\times\gamma^{\mu},\quad \left\{\Gamma^M,\Gamma^N\right\}=2\eta^{MN}. \tag{14}$$

For the case of $d=3$ we take the following explicit form for the SO$(2,1)$ gamma matrices

$$\gamma^{\mu}=\left(i\sigma_2,\sigma_1,\sigma_3\right) \tag{15}$$

The group $G=$OSp$(M/2N)$ is characterized by $g\in G$ of the form

$$g^{-1}=\hat{C}g^{st}\hat{C}^{-1} \tag{16}$$

where g^{st} is the supertranspose of g and $\hat{C}$ is the metric of OSp$(M/2N)$. In supermatrix notation we can write

$$\hat{C} = \begin{pmatrix} 1_M & 0 \\ 0 & C_{2N} \end{pmatrix}, \quad g = \begin{pmatrix} \alpha & \tilde{f}_1 \\ f_2 & A \end{pmatrix}, \quad g^{st} = \begin{pmatrix} \alpha^t & f_2^t \\ -\tilde{f}_1^t & A^t \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} \alpha^t & \tilde{f}_2 \\ f_1 & \tilde{A} \end{pmatrix} \tag{17}$$

with the definitions

$$\tilde{f} = f^t C_{2N}^{-1}, \quad \tilde{A} = C_{2N} A^t C_{2N}^{-1}. \tag{18}$$

Here C_{2N} is the antisymmetric metric for Sp$(2N)$, with properties

$$C_{2N}^t = -C_{2N} = C_{2N}^{-1}, \quad (C_{2N})^2 = -1_{2N}. \tag{19}$$

Note the extra minus sign in $-\tilde{f}_1^t$ in the definition of supertranspose g^{st}. This is necessary so that the supertranspose operation has the property $(g_1 g_2)^{st} = g_2^{st} g_1^{st}$. The necessity of the extra minus sign in $-\tilde{f}^t$ can be traced to the extra minus sign that arises from anticommuting two fermions under transposition. Thus the parameters are constrained by the relation

$$1 \;=\; g^{-1} g = \begin{pmatrix} \alpha^t & \tilde{f}_2 \\ f_1 & \tilde{A} \end{pmatrix} \begin{pmatrix} \alpha & \tilde{f}_1 \\ f_2 & A \end{pmatrix} \tag{20}$$

$$\alpha^t \alpha + \tilde{f}_2 f_2 \;=\; 1_N, \quad \alpha^t \tilde{f}_1 + \tilde{f}_2 A = 0, \quad f_1 \alpha + \tilde{A} f_2 = 0, \quad f_1 \tilde{f}_1 + \tilde{A} A = 1. \tag{21}$$

For infinitesimal group parameters the solution of these constraints are

$$\delta\alpha^t = -\delta\alpha, \quad \delta f_1 = -\delta f_2 \equiv \delta f, \quad \delta\tilde{A} = -\delta A \tag{22}$$

Thus, one may write

$$g = \exp \begin{pmatrix} \delta\alpha & -\delta\tilde{f} \\ \delta f & \delta A \end{pmatrix} \tag{23}$$

where δf is any $M \times (2N)$ fermionic matrix, $\delta\alpha$ is any $M \times M$ antisymmetric matrix and δA is any $(2N) \times (2N)$ symplectic matrix that satisfies $C_{2N} \delta A^t C_{2N}^{-1} = -\delta A$.

4.1 The global current

Using Noether's theorem one finds that the global OSp$(N|2M)$ current of our model is

$$J = g^{-1} L g. \tag{24}$$

where $L = \frac{1}{2} L_{MN} \Gamma^{MN}$. At the classical level the current satisfies

$$J^2 = g^{-1} L g g^{-1} L g = g^{-1} L^2 g \sim 0. \tag{25}$$

The vanishing is because of the Sp$(2, R)$ constraints $X^2 = P^2 = X \cdot P = 0$ at the classical level. As discussed elsewhere [1] the treatment of constraints at the quantum level modifies this result

such that the Casimir operators $\mathrm{Str}(J^n)$ computed from the global currents are all non-zero but fixed at definite constants that define a specific representation.

In the supermatrix notation given above the current is given by

$$J=\begin{pmatrix}\alpha^t & \tilde{f}_2\\ f_1 & \tilde{A}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & L\end{pmatrix}\begin{pmatrix}\alpha & \tilde{f}_1\\ f_2 & A\end{pmatrix}=\begin{pmatrix}\tilde{f}_2Lf_2 & \tilde{f}_2LA\\ \tilde{A}Lf_2 & \tilde{A}LA\end{pmatrix} \tag{26}$$

The parameters α have dropped out because of the internal gauge symmetry. We will now choose further gauges that completely eliminate all degrees of freedom from X^M, P^M, shifting the dynamics to the degrees of freedom in $g(\tau)$. We will show that the remaining degrees of freedom are supertwistors. Using the local $\mathrm{SO}(d,2)$ and $\mathrm{Sp}(2,R)$ gauge symmetries we can map the two vectors $X^M(\tau), P^M(\tau)$ to the constants $X^{+'}=1$, $P^-=1$ with all other components zero $X^M=\delta^M_{+'}$, $P^M=\delta^M_-$. These satisfy the constraints $X^2=P^2=X\cdot P=0$. In this gauge we have

$$L=\frac{1}{2}\Gamma^{MN}L_{MN}=\Gamma^{-'}\Gamma^{+}=\tau^{-}\times\gamma^{+}=\begin{pmatrix}0 & 0\\ \gamma^{+} & 0\end{pmatrix}, \tag{27}$$

Plugging this form into the Lagrangian we find that the remaining degrees of freedom are described by the following dynamics

$$\pounds\sim Tr\left(\tilde{A}\tau^{-}\gamma^{+}\partial_\tau A+\tilde{f}_2\tau^{-}\gamma^{+}\partial_\tau f_2\right). \tag{28}$$

The current given above satisfies the commutation rules of $\mathrm{OSp}(N|2M)$ if we take the basic commutation rules

$$\left\{f^i_\alpha, f^j_\beta\right\}=i\delta^{ij}\hat{L}_{\alpha\beta},\quad \left[A_\alpha^{\ \gamma}, A_\beta^{\ \delta}\right]=i\hat{L}_{\alpha\beta}\left(C^{-1}\right)^{\gamma\delta} \tag{29}$$

where $\hat{L}$ is defined by the relation $L\hat{L}L=L$, and is given by

$$\hat{L}=\tau^{+}\times\gamma^{-}. \tag{30}$$

In the basis of gamma matrices we have chosen, the charge conjugation matrix is $C_4=\tau_1\times C_2$. Using this C_4 we also parametrize the 4-column f_2 and the $4{\times}4$ matrix A in terms of 2 dimensional blocks

$$f_2^i=\begin{pmatrix}\xi^i\\ \chi^i\end{pmatrix},\quad A=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\quad \tilde{f}_{2i}=\begin{pmatrix}\tilde{\chi}_i & \tilde{\xi}_i\end{pmatrix},\quad \tilde{A}=\begin{pmatrix}\tilde{d} & \tilde{b}\\ \tilde{c} & \tilde{a}\end{pmatrix} \tag{31}$$

where $\tilde{\xi}_i=\xi_i^TC_2^{-1}$ and $\tilde{a}=C_2a^TC_2^{-1}$, etc. We may now compute the Lagrangian and the global $\mathrm{OSp}(8|4)$ currents more explicitly

$$\pounds\ \sim\ \frac{1}{2}\tilde{\xi}^i\gamma^+\partial_\tau\xi^j+\frac{1}{2}Tr\left(\tilde{b}\gamma^+\partial_\tau a+\tilde{a}\gamma^+\partial_\tau b\right) \tag{32}$$

$$J\ =\ \begin{pmatrix}\tilde{\xi}^i\gamma^+\xi^j & \tilde{\xi}^i\gamma^+a & \tilde{\xi}^i\gamma^+b\\ \tilde{b}\gamma^+\xi^j & \tilde{b}\gamma^+a & \tilde{b}\gamma^+b\\ \tilde{a}\gamma^+\xi^j & \tilde{a}\gamma^+a & \tilde{a}\gamma^+b\end{pmatrix}=\begin{pmatrix}I^{ij} & -\tilde{Q}^i & -\tilde{S}^i\\ S^j & J & K\\ Q^i & P & -\tilde{J}\end{pmatrix} \tag{33}$$

The canonical pairs are easily determined from the Lagrangian. The currents are quadratic in the canonical pairs. They are reminiscent of the oscillator formalism for supergroups [18] and indeed we will make this connection much clearer in the following paragraphs. The interpretation of the various components of the currents are as follows. The 8×8 block $I^{ij} = \tilde{\xi}^i \gamma^+ \xi^j$ corresponds to the generators of SO(8) , The 2×8 blocks $Q^i = \tilde{a}\gamma^+\xi^i$ and $S^i = \tilde{b}\gamma^+\xi^i$ represent 8 supercharges and 8 superconformal charges respectively. The 4×4 block including J, P, K represents Sp(4) = $SO(3,2)$, with J, P, K corresponding to Lorentz transformations and dilatations, translations and conformal transformations respectively.

Note that the parameters χ, c, d have dropped out from the Lagrangian and the physical global currents. This is because of the extended local kappa supersymmetry and bosonic gauge symmetries. The remaining parameters ξ, a, b are constrained by $gg^{-1} = 1$, or $f_2\tilde{f}_2 + A\tilde{A} = 1$ or

$$\xi^i\tilde{\xi}^i + a\tilde{b} + b\tilde{a} = 0. \tag{34}$$

This also guarantees $J^2 = 0$. Furthermore, because

$$\gamma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{35}$$

is a projection operator, the currents depend only on a few of the parameters in the 2×2 matrices a, b and column matrix ξ. This too is because of the extended gauge symmetry. To further simplify the expressions we define the entries in each matrix

$$\xi^i = \begin{pmatrix} \lambda^i \\ \theta^i \end{pmatrix}, \quad a = \begin{pmatrix} a_3 & a_4 \\ a_1 & a_2 \end{pmatrix}, \quad b = \begin{pmatrix} b_3 & b_4 \\ b_1 & b_2 \end{pmatrix},$$

We find that the currents are given by the unrestricted parameters $(\theta^i, a_1, a_2, b_1, b_2)$ corresponding to the second row of the matrices above, while the remaining parameters drop out from the Lagrangian and currents. We will write $a_\alpha = (a_1, a_2)$, $a^\alpha = (a_2, -a_1)$, raising/lowering indices α by using the Levi-Civita symbol $\varepsilon^{\alpha\beta}$ which is the charge conjugation matrix. In this notation the Lagrangian and currents take the form

$$\pounds \sim \frac{1}{2}\theta^i\dot{\theta}^i + b^\alpha \dot{a}_\alpha, \quad J = \begin{pmatrix} \theta^i\theta^j & \theta^i a_\beta & \theta^i b_\beta \\ b^\alpha\theta^j & b^\alpha a_\beta & b^\alpha b_\beta \\ a^\alpha\theta^j & a^\alpha a_\beta & a^\alpha b_\beta \end{pmatrix} = \begin{pmatrix} I^{ij} & -\tilde{Q}^i & -\tilde{S}^i \\ S^j & J & K \\ Q^i & P & -\tilde{J} \end{pmatrix} \tag{36}$$

The basic non-zero commutation rules

$$\left\{\theta^i, \theta^j\right\} = i\delta^{ij}, \quad \left[a_\alpha, b^\beta\right] = i\delta_\alpha^\beta \tag{37}$$

determine the algebra of the conserved charges J. It is easily seen that the charges form the OSp(8|4) superalgebra. In fact the form of this construction is identical to the oscillator construction of superalgebras [18]. In the present case the real fermions θ^i are SO(8) spinors and

the canonical conjugates a_α, or b^α are real Sp(2) spinors, where Sp(2) $= SO(2,1)$ is the Lorentz subgroup of the conformal group Sp(4) $= SO(3,2)$. In the usual oscillator construction of [18] one chooses the maximal compact subgroup of OSp(8|4), which is SU(4)$\times SU(2)\times (U(1))^2$, and takes a complex quartet of fermionic oscillators in the fundamental representation of SU(4)$\times U(1)$ and a complex doublet of bosonic oscillators in the compact subgroup SU(2)$\times U(1)$, plus their hermitian conjugates. The relation between these two is just a change of basis such that the Sp(4) quartet given in the Sp(2) basis (a_1, a_2, b_1, b_2) is rexpressed in terms of the oscillators $A_\alpha = (a_\alpha + ib^\alpha)/\sqrt{2}$ as a Sp(4) quartet in an SU(2)$\times U(1)$ basis $\left(A_1,\ A_2,\ A_2^\dagger, -A_1^\dagger\right)$. Similarly the real SO(8) spinor θ^i is re-expressed in the complex SU(4)$\times U(1)$ basis in terms of fermionic oscillators. This construction proves the unitarity of the representation, and identifies the super conformal particle with the super doubleton representation of OSp(8|4). It is evident that the same general arguments hold for OSp(N|4).

Returning to the Sp(2) $= SO(3,2)$ spinors a_α or b^α, we can interpret them as the twistor representation of the particle canonical coordinates x^μ, p^μ à la Penrose as follows. To avoid complications due to quantum ordering, we will discuss only the classical version of this interpretation. Also, we concentrate only on the purely bosonic Sp(4) to avoid complications with the supergroup. Starting with the generators $P^\alpha_\beta = a^\alpha a_\beta$ which form a traceless 2×2 matrix, we identify the momentum as

$$a^\alpha a_\beta = (\gamma^\mu)^\alpha_\beta\, p_\mu\,, \tag{38}$$

Evidently it describes a massless particle, since $p^2\delta^\alpha_\beta = a^\alpha a_\gamma a^\gamma a_\beta = 0$. The coordinate x^μ is defined by the following relation between the two spinors a_α, b_β

$$b_\alpha = x^\mu\,(\gamma_\mu)^\beta_\alpha\, a_\beta. \tag{39}$$

To show that this is indeed the case, consider the conformal generator K^μ written in the form $K^\alpha_\beta = K^\mu(\gamma^\mu)^\alpha_\beta = b^\alpha b_\beta$ and insert the expression for b_α

$$K^\mu = \frac{1}{2}b^\alpha(\gamma^\mu)^\beta_\alpha\, b_\beta = -\frac{1}{2}x^\lambda a^\alpha(\gamma_\lambda\gamma^\mu\gamma_\nu)^\beta_\alpha\, a_\beta x^\nu \tag{40}$$

$$= -\frac{1}{2}x^\lambda p_\sigma Tr(\gamma^\sigma\gamma_\lambda\gamma^\mu\gamma_\nu)^\beta_\alpha\, x^\nu = \frac{1}{2}x^2 p_\mu - x\cdot p x^\mu. \tag{41}$$

This result is the well known expression for the conformal generator (avoiding quantum ordering, or fermionic contributions). Similarly we compute the dimension operator $D = \frac{1}{2}b^\alpha a_\alpha$ and the Lorentz generator $J^{\mu\nu} = b^\alpha(\gamma^{\mu\nu})^\beta_\alpha\, a_\beta$

$$D = \frac{1}{2}b^\alpha a_\alpha = \frac{1}{2}x^\mu a^\alpha(\gamma_\mu)^\beta_\alpha\, a_\beta = \frac{1}{2}x^\mu p^\nu Tr(\gamma_\mu\gamma_\nu) = x\cdot p \tag{42}$$

$$J^{\mu\nu} = b^\alpha(\gamma^{\mu\nu})^\beta_\alpha\, a_\beta = x^\lambda a^\alpha(\gamma_\lambda\gamma^{\mu\nu})^\beta_\alpha\, a_\beta = x^\lambda p_\sigma Tr(\gamma_\lambda\gamma^{\mu\nu}\gamma_\sigma) = x_\mu p_\nu - x_\nu p_\mu \tag{43}$$

These are the correct expressions for the massless bosonic particle.

This makes it evident that the basis we have defined in terms of the super variables $(\theta^i, a_\alpha, b_\alpha)$ is indeed the twistor basis for the conformal superparticle. In the presence of fermions the relation

between the spinors b_α, a_α is considerably more complicated such that the correct generators K^μ, $J^{\mu\nu}$ emerge, as given in [4]. The generalized relation including the fermions is obtained by applying the kappa and other gauge transformations that take the model from the fixed gauge $g(\tau) = t(\tau)$ used in [4] to the fixed gauge of the twistor formalism described in this paper.

5 Discussion and generalizations

From the previous section, it is evident that the approach is applied in a straightforward manner in $d = 4, 6$ using the supergroups $SU(2,2|N)$ and $\mathrm{OSp}(8^*|N)$ respectively. The main point is the shifting of the degrees of freedom from X, P to g or vice-versa by using the gauge symmetries $\mathrm{Sp}(2,R)$ and $\mathrm{SO}(d,2)$. When X, P are eliminated, the remaining gauge symmetries reduce further the degrees of freedom in g to the super twistor variables for those dimensions. In particular, for $d = 4$ we find agreement with [20]. Similarly, twistors can be gauge transformed to coordinate representation to describe the same system. Twistors were invented as a means of describing conformal systems covariantly in linear realizations. We have now shown that they are related to another linear realization, namely Lorentz transformations in 2T physics.

In this paper we have mainly concentrated on a specific $\mathrm{Sp}(2,R)$ gauge choice, namely $X^{+'} = 1$, $P^{+'} = 0$ which relates to the $\mathrm{SO}(d-1,1)$ covariant massless relativistic particle. This is a particular embedding of d dimensions inside $d+2$ dimensions, and corresponds to a particular 1T physics interpretation of the 2T theory. As we know from [1]-[9] there are many other 1T embeddings with different 1T physics interpretations. Such other $\mathrm{Sp}(2,R)$ gauges may now be combined with the present techniques of shifting particle variables to twistor-like variables embedded in g, and thus find new twistor-like realizations of 1T dynamical systems, as well as establish duality-type relations among them.

The reader will notice that the spacetime and internal subgroups of $\mathrm{OSp}(8|4)$, $\mathrm{SU}(2,2|4)$, $\mathrm{OSp}(8^*|4)$ were treated in an asymmetric manner in the coupling introduced in (7). Since these supergroups describe the supersymmetries in $\mathrm{AdS}_7 \times S_4$, $\mathrm{AdS}_5 \times S_5$, $\mathrm{AdS}_4 \times S_7$, respectively, one may wonder if there is a more symmetric treatment of the AdS×S spaces that would apply to these cases. In fact, in addition to the spacetime X^M, P^M phase space we may introduce internal Y^I, K^I phase space, define the internal angular momentum $L^{IJ} = Y^I K^J - Y^J K^I$ and couple it to the supergroup Cartan connection in the same manner as (7). When both L^{MN} and L^{IJ} have non-zero coupling it is possible to maintain a kappa-type local supersymmetry as well as the $\mathrm{Sp}(2,R)$ local symmetry coupled to all the coordinates $\left(X^M, Y^I\right)$ and momenta $\left(P^M, K^I\right)$. This variation leads to more interesting and intricate 2T models. In fact it is even possible to couple to any a subgroup of the internal group $\mathrm{SO}(N)$, $\mathrm{SU}(N)$, $\mathrm{Sp}(N)$ that appear in (6). If only a subgroup H is gauged, then only the corresponding degrees of freedom can be removed. The remaining coset plays the role of harmonic superspace recently discussed in [21].

As mentioned earlier, a motivation for developing these techniques was the study of the quantum system for the toy M-model that will be discussed elsewhere [9]. Having shown that the

approach works and establishes connections among previously better understood systems, it may now be used to explore new systems.

References

[1] I. Bars, C. Deliduman and O. Andreev, Phys. Rev. **D58** (1998) 066004, or hep-th/9803188; I. Bars, Phys. Rev. **D58** (1998) 066006, or hep-th/9804028; S. Vongehr, "Examples of black holes in two time physics", hep-th/9907077; for a review see, I. Bars, hep-th/9809034.

[2] I. Bars, Phys. Rev. **D59** (1999) 045019, or hep-th/9810025.

[3] I. Bars and C. Deliduman, Phys. Rev. **D58** (1998) 106004, or hep-th/9806085.

[4] I. Bars, C. Deliduman and D. Minic, Phys. Rev. **D59** (1999) 125004 or hep-th/9812161.

[5] I. Bars, C. Deliduman and D. Minic, Phys. Lett. **B457** (1999) 275 or hep-th/9904063.

[6] I. Bars, C. Deliduman and D. Minic, Phys. Lett. **B466** (1999) 135 or hep-th/9906223.

[7] I. Bars, "Two-time physics with gravitational and gauge field backgrounds", hep-th/0002140.

[8] I. Bars, "Two-time physics in Field Theory", hep-th/0003100.

[9] I. Bars, "A toy-M-model", in preparation.

[10] P.A.M Dirac, Ann. Math. **37** (1936) 429.

[11] H. A. Kastrup, Phys. Rev. **150** (1966) 1183.

[12] G. Mack and A. Salam, Ann. Phys. **53** (1969) 174.

[13] C. R. Preitschopf and M. A. Vasilev, hep-th/9812113.

[14] R. Marnelius, Phys. Rev. **D20** (1979) 2091; W. Siegel, Int. J. Mod. Phys. **A3** (1988) 2713.

[15] I. Bars and C. Kounnas, Phys. Lett. **B402** (1997) 25; Phys. Rev. **D56** (1997)3664; I. Bars and C. Deliduman, Phys. Rev. **D56** (1997) 6579 or hep-th/9707215; Phys. Lett. **B417** (1998)24 or hep-th/9710066; E. Sezgin, Phys. Lett. **B403** (1997) 265; E. Sezgin and Rudychev, hep-th/9711128; Phys. Lett. **B424** (1998) 60.

[16] R. Penrose, J. Math. Phys. **8** (1967) 345; Rep. Math. Phys. **12** (1977) 65; R. Penrose and M.A.H. Mac Callum, Phys. Rep. **6** (1972) 241; R. Penrose and W. Rindler, Spinors and Space-Time, V.1,2. Cambridge Univ. Press 1986.

[17] A. Ferber, Nucl. Phys. **B132** (1977) 55; I. Bandos, J. Lukierski , C. Preitschopf, D. Sorokin, Phys. Rev. **D61** (2000) 065009, and references therein.

[18] I. Bars and M. Günaydin, Comm. Math. Phys. **91** (1983) 31. M. Günaydin and N. Marcus, Class. and Quant. Grav. **2** (1985) L11; M. Günaydin, D. Minic and M. Zagermann, Nucl. Phys. **B534** (1998) 96; Nucl. Phys. **B544** (1999) 737. M. Günaydin and D. Minic, Nucl. Phys. **B523** (1998) 145.

[19] J. Schwarz, Nucl. Phys. **B185** (1981) 221.

[20] P. Claus, M. Günaydin, R. Kallosh, J. Rahmfeld and Y. Zunger, hep-th/9905112; P. Claus, J. Rahmfeld and Y. Zunger, Phys. Lett. **B466** (1999) 181.

[21] S. Ferrara and E. Sokatchev, "Short representations of $SU(2,2|N)$ and harmonic superspace analyticity", hep-th/9912168. A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quantum Grav. **1** (1984) 498 and **2** (1985) 155.